BUBBLE STABILIZATION OF CHEBYSHEV-LEGENDRE HIGH-ORDER ELEMENT METHODS FOR THE ADVECTION-DIFFUSION EQUATION

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Abstract. The bubble stabilization technique of Chebyshev-Legendre high-order element methods for one dimensional advection-diffusion equation is analyzed for the proposed scheme by Canuto and Puppo in [8]. We also analyze the finite element lower-order preconditioner for the proposed stabilized linear system. Further, the numerical results are provided to support the developed theories for the convergence and preconditioning.

1. Introduction

In 1990's the successful bubble stabilization method combined with Legendre-Galerkin scheme was reported in [8] for a stationary advection-diffusion problem to control spurious oscillations occurred in the Legendre polynomial approximation (see [6, 7] for example). This method is known as an adaptation of SUPG-stabilization (see [2, 3] for example) for Legendre-Galerkin spectral methods (see [4, 7, 8] for example). The key ideas in [5, 8] for bubble stabilization to suppress such oscillations are to combine the accuracy of Legendre interpolation polynomials with the flexibility of local low-order finite elements with the help of the uniform lower-order/high-order interpolation properties (see [6, 7, 8] and etc.). Unfortunately, such a successful technique has not been applied to the Chebyshev-Legendre-Galerkin scheme up to now. Hence, the primal goal of this paper is to adopt the same bubble-aided formulation in [8] to Chebyshev-Legendre-Galerkin scheme.

By contrast with the bubble stabilization of Legendre spectral methods, a non-realistic difficulty is arisen in the bubble stabilization of Chebyshev spectral methods because it may not deal with a boundary layer problem if one uses the Chebyshev-Gauss-Lobatto(:=CGL) nodes and Chebyshev weight functions (see [12]). This is due to the Chebyshev weight function which does not allow the comparison of the piecewise linear functions with its Chebyshev interpolation polynomials using CGL nodes in the sense of the standard L^2 and H^1 norms without Chebyshev weight function. Because of such a difficulty, the Legendre spectral methods with bubble stabilization results in [5, 8] have not been extended to the Chebyshev-Legendre case for a convection-diffusion problem (4.1).

Hence, owing to both approximation results for Chebyshev interpolation polynomials in [13] and the lower-order/high-order interpolation property in [10], it is possible for us to apply the developed same structures in [8] to the Chebyshev-Galerkin polynomial approximation using the CGL points. Then we can provide the stability and spectral convergence theory in terms of non-weighted Sobolev norms. For evidences the H^1 norm errors are computed for a model problem.

Owing to several successful computational results in [8] about control of the condition numbers by the lower-order finite element preconditioner, one may provide some analysis on such a preconditioner using the results in [9]. However, for the present proposed scheme we provide an analysis for a lower-order preconditioner which controls the condition numbers for such a scheme. In fact, we show that the condition numbers are independent of both mesh sizes and the degrees of Chebyshev polynomials employed. Several numerical results are provided to support these theoretical phenomena.

The outline of this paper is as follows: Some notations and definitions are provided in Section 2. In Section 3, some properties of Chebyshev interpolation operator are analyzed in terms of non-weighted Sobolev norms. Further, uniform bounds of the global interpolation polynomials with respect to the global piecewise linear functions are provided in the non-weighted H^1-Sobolev norm. Following the same way in [8], the bubble-stabilized Chebyshev method is described in Section 4 and the stability and convergence analysis by mentioning the necessary details only are shown with numerical results in Section 5. But for a general case one has to refer to [8]. In Section 6, we provide a complete analysis of lower-order preconditioner with computational evidences. Finally, we add some conclusions in last section.

2. Notations and definitions

Throughout this paper, we use the standard function spaces and related norms such as the usual L^2 product $(\cdot, \cdot)_D$ and its norm $\| \cdot \|_D^2 = (\cdot, \cdot)_D$, the standard Sobolev spaces H_0^1 and $H^k(D)$ and its norm $\| \cdot \|_{H^k(D)}$ and semi-norm $| \cdot |_{H^k(D)}$ where D is a given domain. The uniform knots of $\Lambda := [-1, 1]$ are

denoted as $\{t_k\}_{k=0}^M$ ordered by $-1 =: t_0 < t_1 < \cdots < t_{M-1} < t_M := 1$ with a mesh size $\tau := t_j - t_{j-1} = 2/M$. The subinterval E_j of Λ is denoted as $E_j := [t_{j-1}, t_j]$ for $j = 1, 2, \ldots, M$.

For usages of Chebyshev-Legendre spectral element methods, we need the reference CGL nodes $\{\xi_k^c\}_{k=0}^N$ given by $\xi_k^c := -\cos\frac{\pi k}{N}$ and LGL nodes $\{\xi_k^l\}_{k=0}^N$ which are the zeros of $(1-t^2)L_N'(t)$, where L_N is the Nth Legendre polynomial defined in Λ. On the subinterval E_j, we denote $\eta_{k+N(j-1)}^q$ as the kth CGL nodes for $q = c$ or LGL nodes for $q = l$ such that $\eta_{k+N(j-1)}^q =: \eta_{j,k}^q$ where

$$\eta_{j,k}^q = \frac{\tau}{2}\xi_k^q + \frac{1}{2}(t_{j-1} + t_j), \quad j = 1, \ldots, M, \; k = 0, \ldots N.$$

Let $\Lambda_\nu = [\eta_{\nu-1}^c, \eta_\nu^c]$, $h_\nu = \eta_\nu^c - \eta_{\nu-1}^c$ $(\nu = 1, \ldots, NM)$. Denote $\Lambda_k^j = [\eta_{j,k-1}^c, \eta_{j,k}^c]$ as k^{th} subinterval of the interval E_j with its size $h_k^j := \eta_{j,k}^c - \eta_{j,k-1}^c$. We will use N to denote a fixed degree N of Chebyshev or Legendre polynomial used from now on. For the reference polynomial space $\mathcal{P}_N$ on the interval Λ, the Chebyshev-Lagrange basis functions $\{\hat{\phi}_i(t)\}_{i=0}^N$ are employed which satisfy

$$\hat{\phi}_i(\xi_k^c) = \delta_{i,k} \quad \text{for} \quad i, k = 0, 1, \ldots, N,$$

where δ denotes the Kronecker delta function. Let $\mathcal{P}_N^\tau$ be the subspace of $C(\Lambda)$ which consists of the piecewise continuous polynomials of degree N defined on the subinterval E_j whose basis is given by the piecewise continuous Chebyshev-Lagrange polynomial $\{\phi_\mu\}_{\mu=0}^{MN}$ defined appropriately (see the detailed description in [10]). Let $\mathcal{F}_h$ be the reference space consisting of all piecewise continuous linear functions $\{\hat{\psi}_i(t)\}$ defined on Λ satisfying $\hat{\psi}_i(\xi_k^c) = \delta_{i,k}$ and let $\mathcal{F}_h^\tau$ be the space of all piecewise continuous linear functions $\{\psi_\mu\}_{\mu=0}^{MN}$ such that $\psi_\mu(\eta_\nu^c) = \delta_{\mu,\nu}$, $\mu, \nu = 0, 1, \ldots MN$ defined appropriately like $\mathcal{P}_N^\tau$.

3. Properties of interpolation polynomials

Define $\mathcal{I}_N : C(\Lambda) \to \mathcal{P}_N$ as the reference interpolation operator such that

$$(\mathcal{I}_N u)(\xi_k^c) = u(\xi_k^c), \quad k = 0, 1, \ldots, N, \quad \text{for all} \quad u \in C(\Lambda),$$

and let $\mathcal{I}_N^j$ be the interpolation operator from $C(E_j)$ to a polynomial space $\mathcal{P}_N$ on E_j using the local CGL nodes $\{\eta_{j,k}^c\}$ in the jth subinterval E_j such that

$$(\mathcal{I}_N^j u)(\eta_{j,k}^c) = u(\eta_{j,k}^c), \quad k = 0, 1, \ldots, N, \quad \text{for all} \quad u \in C(E_j).$$

The global interpolation operator $\mathcal{I}_N^\tau : C(\Lambda) \to \mathcal{P}_N^\tau$ is defined such that for all $u_h^\tau \in C(\Lambda)$

$$u_N^\tau(\eta_\nu^c) := \left(\mathcal{I}_N^\tau u_h^\tau\right)(\eta_\nu^c) = u_h^\tau(\eta_\nu^c), \quad \nu = 0, 1, \ldots, MN$$

whose inverse interpolation operator $\mathcal{J}_h^\tau$ from $\mathcal{P}_N^\tau$ to $\mathcal{F}_h^\tau$ satisfies

$$\mathcal{J}_h^\tau u_N^\tau = u_h^\tau, \quad \text{and} \quad u_N^\tau = \mathcal{I}_N^\tau u_h^\tau \quad \text{for all} \quad u_h^\tau \in \mathcal{F}_h^\tau.$$

Hence the interpolant u_N^τ is the function of $\mathcal{P}_N^\tau$ whose restriction to the each subinterval E_j interpolates the function u at $N+1$ local CGL nodes in E_j.

Lemma 3.1. *For $u \in H^m(\Lambda)$ where $m \geq 1$, it follows that*

$$(3.1) \qquad \|\mathcal{I}_N u - u\|_{H^k(\Lambda)} \leq C_1 N^{k-m} |u|_{H^m(\Lambda)}, \quad k = 0, 1,$$

where C_1 is an absolute constant independent of N.

Proof. This proof in terms of norm $\|u\|_{H^m(\Lambda)}$ is found in Lemma 3.3 of [13]. Following the proof of Lemma 3.3 of [13], one may see that this $H^m(\Lambda)$ norm can be replaced by $H^m(\Lambda)$ semi-norm using the result of the Legendre interpolation operator $\mathcal{I}_N^L$ at the $N+1$ LGL nodes in Λ such that

$$\|\mathcal{I}_N^L u - u\|_{H^k(\Lambda)} \leq C N^{k-m} |u|_{H^m(\Lambda)}$$

found in [7] (see formulas (5.4.33) and (5.4.35) therein). $\qquad \square$

For the Legendre interpolation operator using LGL nodes designed for SEM, its interpolation error analysis can be found in [6] (see Formula (5.4.3) and Section 5.4.3 for proof therein). However, we provide such a similar result for the Chebyshev interpolation operator $\mathcal{I}_N^\tau$ using CGL nodes in terms of non-weighted Sobolev norm.

Proposition 3.2. *For all $\phi \in H^m(\Lambda)$, $m \geq 1$, it follows that*

$$(3.2) \qquad \|\phi - \mathcal{I}_N^\tau \phi\|_{H^k(\Lambda)} \leq C_1 N^{k-m} \left(\frac{\tau}{2}\right)^{m-k} |\phi|_{H^m(\Lambda)} \quad for \quad k = 0, 1,$$

where C_1 is the same constant as in (3.1) independent of N and meshsize τ.

Proof. First of all, let us recall

$$(3.3) \quad |\hat{\phi}|_{H^m(\Lambda)} = \left(\frac{\tau}{2}\right)^{m-\frac{1}{2}} |\phi|_{H^m(E_j)}, \ (m \geq 1) \ and \ \|\hat{\phi}\|_\Lambda = \left(\frac{\tau}{2}\right)^{-\frac{1}{2}} \|\phi\|_{E_j},$$

where $\hat{\phi}$ is the transformed reference function defined on Λ from the function ϕ defined on E_j. Then, using (3.3) and (3.1),

$$|\phi - \mathcal{I}_N^j \phi|_{H^1(E_j)}^2 = \left(\frac{2}{\tau}\right) |\hat{\phi} - \mathcal{I}_N \hat{\phi}|_{H^1(\Lambda)}^2 \leq C_1^2 \left(\frac{2}{\tau}\right) N^{2(1-m)} |\hat{\phi}|_{H^m(\Lambda)}^2,$$

$$\|\phi - \mathcal{I}_N^j \phi\|_{L^2(E_j)}^2 = \left(\frac{\tau}{2}\right) \|\hat{\phi} - \mathcal{I}_N \hat{\phi}\|_{L^2(\Lambda)}^2 \leq C_1^2 \left(\frac{\tau}{2}\right) N^{-2m} |\hat{\phi}|_{H^m(\Lambda)}^2.$$

Using (3.3), it follows that

$$|\phi - \mathcal{I}_N^j \phi|_{H^1(E_j)}^2 \leq C_1^2 N^{2(1-m)} \left(\frac{\tau}{2}\right)^{2(m-1)} |\phi|_{H^m(E_j)}^2,$$

$$\|\phi - \mathcal{I}_N^j \phi\|_{L^2(E_j)}^2 \leq C_1^2 N^{-2m} \left(\frac{\tau}{2}\right)^{2m} |\phi|_{H^m(E_j)}^2.$$

Therefore, combining all of these yields to

$$\|\phi - \mathcal{I}_N^j \phi\|_{H^k(E_j)}^2 \leq C_1^2 N^{2(k-m)} \left(\frac{\tau}{2}\right)^{2(m-k)} |\phi|_{H^m(E_j)}^2$$

which leads to

$$\|\phi - \mathcal{I}_N^\tau \phi\|_{H^k(\Lambda)}^2 = \sum_{j=1}^N \|\phi - \mathcal{I}_N^j \phi\|_{H^k(E_j)}^2 \leq C_1^2 N^{2(k-m)} \left(\frac{\tau}{2}\right)^{2(m-k)} |\phi|_{H^m(\Lambda)}^2.$$

This argument completes the proof. □

It is known that the uniform lower-order/high-order interpolation property holds for the Legendre interpolation polynomials in terms of the Legendre weight $w(x) = 1$ (see [7] for example) and for the Chebyshev interpolation polynomials in terms of the Chebyshev weight function $w(x) = \frac{1}{\sqrt{1-x^2}}$ in [11]. To see how the bubble-aided Chebyshev-Legendre-Galerkin method works for the same physical phenomenal problems such as boundary layer problems dealt with the bubble-aided Legendre-Galerkin method, it is required to compare piecewise linear functions with Chebyshev polynomials in terms of the H^1-norm (semi-norm) and L^2-norm.

Proposition 3.3. *It follows that for all* $u_h^\tau \in \mathcal{F}_h^\tau$

(3.4)
$$C_2 \|u_h^\tau\|_\Lambda \leq \|\mathcal{I}_N^\tau u_h^\tau\|_\Lambda \leq C_N \|u_h^\tau\|_\Lambda$$

and

(3.5)
$$C_2 \|u_h^\tau\|_{H^1(\Lambda)} \leq \|\mathcal{I}_N^\tau u_h^\tau\|_{H^1(\Lambda)} \leq C_3 \|u_h^\tau\|_{H^1(\Lambda)},$$
$$C_2 |u_h^\tau|_{H^1(\Lambda)} \leq |\mathcal{I}_N^\tau u_h^\tau|_{H^1(\Lambda)} \leq C_3 |u_h^\tau|_{H^1(\Lambda)},$$

where the positive constants C_2 and C_3 do not depend on both the degree N and the mesh size τ, but C_N is a positive constant dependent on N only.

Proof. For the proof of (3.4), see (7.8) in [10]. For the proof of (3.5), see Proposition 4.9 and Theorem 4.10 in [10]. We note that the estimates in (3.5) do not provide the proof of (3.4). □

Note that the constant C_N of the upper bound in (3.4) does depend on the degree N of Chebyshev polynomial, but this bound will not be used to prove the convergence analysis and stability analysis fortunately.

4. Bubble-stabilized high-order element method using CGL nodes

In the same formulation developed in [8], the high-order Galerkin method using CGL nodes will be discussed for the one-dimensional advection-diffusion problem

(4.1)
$$\mathcal{L}u := -\nu u_{xx} + \beta u_x = f \quad \text{in} \quad \Lambda = (-1, 1)$$
$$u(-1) = u(1) = 0,$$

where ν is a positive constant. In order to see the effect of bubble in the formulation of high-order Chebyshev-Legendre Galerkin method, we assume that β in (4.1) is a nonnegative constant even if a general β works for its

stability and convergence analysis as in [8]. Hence, one may see how the bubble-stabilized high-order Galerkin method using LGL nodes works and how its numerical results seem to be.

Let us define the bilinear form $a(\cdot, \cdot)$ on $H_0^1(\Lambda) \times H_0^1(\Lambda)$ as

$$(4.2) \qquad\qquad a(u, v) = \nu \, (u_x, v_x)_\Lambda + (\beta u_x, v)_\Lambda \,.$$

Then the Galerkin formulation for (4.1) can be written as

$$(4.3) \quad \text{find} \quad u \in H_0^1(\Lambda) \quad \text{satisfying} \quad a(u, v) = (f, v)_\Lambda \quad \text{for all} \quad v \in H_0^1(\Lambda).$$

Consider the high-order element method using CGL nodes for (4.3): Find $u_N^\tau \in \mathcal{V}_N^\tau := \mathcal{P}_N^\tau \cap H_0^1(\Lambda)$ such that

$$(4.4) \qquad\qquad a(u_N^\tau, v_N^\tau) = (f, v_N^\tau)_\Lambda \quad \text{for all} \quad v_N^\tau \in \mathcal{V}_N^\tau.$$

It is known that the oscillation behavior of u_N^τ of the solution in (4.4) has been theoretically analyzed in [4] for the one element Legendre Galerkin method in the case of constant coefficient $\beta = 1$ and $f = 1$. In an attempt to suppress such oscillations in spectral method, the bubble stabilized Legendre-Galerkin methods for (4.1) was analyzed theoretically and numerically in [8].

To discuss the bubble stabilization, let the finite dimensional subspace $\mathcal{B}_N$ of $H_0^1(\Lambda)$ be spanned by chosen nonnegative reference quadratic bubble functions $\{\hat{b}_j \in H_0^1(\Lambda)\}_{j=1}^N$ with its support Λ_j because it is known (see [1, 8]) that the maximum value of $c(\hat{b}) = \left(\int_0^1 \hat{b} \, d\hat{x}\right)^2 / \int_0^1 (\hat{b}_{\hat{x}})^2 d\hat{x}$ for $\hat{b} \in H_0^1(\Lambda)$ is achieved by the parabolic bubble $\hat{b}(\hat{x}) = \hat{x}(1 - \hat{x})$. By the linear transformation from the reference bubble $\hat{b}$ on Λ to a bubble function $b_\mu := b_k^j$ on $[\eta_{\mu-1}^c, \eta_\mu^c] := [\eta_{j,k-1}^c, \eta_{j,k}^c]$ which will be used for its extension to 0 outside of $[\eta_{j,k-1}^c, \eta_{j,k}^c]$, we denote the finite dimensional subspace $\mathcal{B}_N^\tau$ as

$$\mathcal{B}_N^\tau := span\{b_\mu \,|\, \mu = 1, \ldots, MN\}.$$

Note 4.1. We mention that the construction of basis functions for the spaces $\mathcal{P}_N^\tau$ and $\mathcal{F}_N^\tau$ should be defined appropriately at each knots t_j for which the basis functions must be 1 at t_j and 0 at other local CGL nodes.

Now, the high-order element Chebyshev-Galerkin method based on the bubble-aided space $\mathcal{W}_N^\tau = \mathcal{V}_N^\tau \bigoplus \mathcal{B}_N^\tau$ is to find $u_{N,b}^\tau \in \mathcal{W}_N^\tau$ satisfying

$$(4.5) \qquad\qquad a(u_{N,b}^\tau, w_{N,b}^\tau) = (f, w_{N,b}^\tau)_\Lambda \quad \text{for all} \quad w_{N,b}^\tau \in \mathcal{W}_N^\tau.$$

The Chebyshev-Galerkin weak formulation (4.5) can be written as a block 2×2 system; Find $u_N^\tau \in \mathcal{V}_N^\tau$ and $u_b^\tau \in \mathcal{B}_N^\tau$ such that

$$(4.6) \qquad a(u_N^\tau, v_N^\tau) + a(u_b^\tau, v_N^\tau) = (f, v_N^\tau)_\Lambda, \quad v_N^\tau \in \mathcal{V}_N^\tau$$
$$a(u_N^\tau, v_b^\tau) + a(u_b^\tau, v_b^\tau) = (f, v_b^\tau)_\Lambda, \quad v_b^\tau \in \mathcal{B}_N^\tau.$$

Let us define the L^2-orthogonal projection $\mathcal{J}_h$ from $L^2(\Lambda)$ onto F_h^τ where F_h^τ as the space of continuous functions whose restriction on Λ_k^j is constant for

$j = 1, \ldots, M$, $k = 1, \ldots, N$. Then, following the reason stated in [8] due to Proposition 3.3, we will consider the following discretized scheme: Find $u_N^\tau \in \mathcal{V}_N^\tau$ and $u_b^\tau \in \mathcal{B}_N^\tau$ such that

$$(4.7) \qquad a(u_N^\tau, v_N^\tau) + a(u_b^\tau, v_h^\tau) = (f, v_N^\tau)_\Lambda, \quad v_N^\tau \in \mathcal{V}_N^\tau$$
$$(J_h(\mathcal{L}u_N^\tau), v_b^\tau)_\Lambda + a(u_b^\tau, v_b^\tau) = (J_h f, v_b^\tau)_\Lambda, \quad v_b^\tau \in \mathcal{B}_N^\tau.$$

The elimination of the bubble component in each Λ_k^j as done in [8] with the same bubble components in each cell Λ_k^j leads to

Find $u_N^\tau \in \mathcal{V}_N^\tau$ such that for all $v_N^\tau \in \mathcal{V}_N^\tau$

$$(4.8) \quad \nu(u_{N,x}^\tau, v_{N,x}^\tau)_\Lambda + \beta(u_{N,x}^\tau, v_N^\tau)_\Lambda - \sum_{j=1}^M \sum_{k=1}^N \Big[\nu \gamma_k^j \beta(u_{N,xx}^\tau, v_{h,x}^\tau)_{\Lambda_k^j} \Big]$$
$$+ \sum_{j=1}^M \sum_{k=1}^N \Big[\gamma_k^j \beta^2(u_{N,x}^\tau, v_{h,x}^\tau)_{\Lambda_k^j} \Big] = (f, v_N^\tau)_\Lambda + \sum_{j=1}^M \sum_{k=1}^N \Big[\gamma_k^j \beta(f, v_{h,x}^\tau)_{\Lambda_k^j} \Big],$$

where

$$(4.9) \qquad \gamma_k^j = c(\hat{b}_k^j) \frac{(h_k^j)^2}{\nu}, \quad c(\hat{b}) = \frac{\left(\int_0^1 \hat{b} \, d\hat{x} \right)^2}{\int_0^1 (\hat{b}_{\hat{x}})^2 d\hat{x}}.$$

Proposition 4.1. *There exists a constant C_4 independent of the degree N and the mesh size τ such that for any polynomial $p_N \in \mathcal{P}_N^\tau$ of degree N*

$$(4.10) \qquad \sum_{j=1}^M \sum_{k=1}^N (h_k^j)^2 \|p_{N,x}\|_{\Lambda_k^j}^2 \leq C_4 \|p_N\|_\Lambda^2,$$

where $h_k^j := \eta_{j,k}^c - \eta_{j,k-1}^c$ and $\Lambda_k^j = (\eta_{j,k}^c, \eta_{j,k-1}^c)$.

Proof. Note that Proposition 3.1 in [8] can be modified on each interval E_j to

$$(4.11) \qquad \sum_{k=1}^N (h_k^j)^2 \|p_{N,x}\|_{\Lambda_k^j}^2 \leq C \|p_N\|_{E_j}^2, \quad j = 1, 2, \ldots, M$$

for all polynomial of degree $N \geq 2$ because the proof of Proposition 3.1 in [8] works for CGL points where C does not depend on the degree N. Now summing up (4.11) from $j = 1$ to M yields the conclusion where the constant does not dependent of N and z. $\square$

Since the uniform mesh size τ and the same degree of polynomials on each interval E_j, it follows that for a given N and M

$$(4.12) \qquad h := \max_{1 \leq j \leq M, 1 \leq k \leq N} h_k^j = \max_{1 \leq k \leq N} h_k^1 = \cdots = \max_{1 \leq k \leq N} h_k^M.$$

5. Stability and convergence

Let us take $v_N^\tau = u_N^\tau$ in (4.8) we have

$$(5.1) \quad \nu\|u_{N,x}^\tau\|^2 - \sum_{j=1}^{M}\sum_{k=1}^{N} \nu\,\gamma_k^j\,\beta\,(u_{N,xx}^\tau, u_{h,x}^\tau)_{\Lambda_k^j} + \sum_{j=1}^{M}\sum_{k=1}^{N} \gamma_k^j\,\beta^2\,(u_{N,x}^\tau, u_{h,x}^\tau)_{\Lambda_k^j}$$

$$\leq (f, u_N^\tau)_\Lambda + \sum_{j=1}^{M}\sum_{k=1}^{N} \gamma_k^j\,\beta\,(f, u_{h,x}^\tau)_{\Lambda_k^j}.$$

By modifying the proof of Theorem 3.1 and Theorem 3.2 in [8] line by line, one may have the stability and convergence theorems immediately. Hence, one should refer to the detailed proof in [8] for general case. But for reader's convenience we provide the necessary details. As suggested in [8], we take the reference bubble function as the parabolic bubble $\hat{b}(\hat{x}) = \hat{x}(1 - \hat{x})$, in $[0, 1]$ so that the value of $c(\hat{b})$ in (4.9) has the maximum value $\frac{1}{12}$. Then, the value of γ_k^j is $\gamma_k^j = \frac{(h_k^j)^2}{12\nu}$.

Theorem 5.1. *Let $u_N^\tau \in \mathcal{V}_N^\tau$ be a solution of (4.8). Assume that*

$$(5.2) \qquad\qquad \max_{j,k} c(\hat{b}_k^j) = \frac{1}{12} \leq \frac{1}{C_4},$$

where C_4 is a positive constant satisfying the inequality (4.10). Then the following estimate holds:

$$(5.3) \qquad\qquad \nu\|u_{N,x}^\tau\|^2 + \beta^2 \sum_{j=1}^{M}\sum_{k=1}^{N} \gamma_k^j\|u_{h,x}^\tau\|_{\Lambda_k^j}^2 \leq \tilde{C}\|f\|^2,$$

where the constant $\tilde{C}$ is $\tilde{C} = O(\nu^{-1})$.

Proof. First, using (4.9) and (4.10), the second term of the left side of (5.1) can be estimated as

$$\left| \sum_{j=1}^{M}\sum_{k=1}^{N} \nu\,\gamma_k^j\,\beta\,(u_{N,xx}^\tau, u_{h,x}^\tau)_{\Lambda_k^j} \right|$$

$$(5.4) \qquad \leq \frac{\epsilon_1}{2} \sum_{j=1}^{M}\sum_{k=1}^{N} \nu^2\gamma_k^j\|u_{N,xx}^\tau\|_{\Lambda_k^j}^2 + \frac{1}{2\epsilon_1} \sum_{j=1}^{M}\sum_{k=1}^{N} \beta^2\gamma_k^j\|u_{h,x}^\tau\|_{\Lambda_k^j}^2$$

$$\leq \frac{\epsilon_1}{2}\frac{\nu}{12}C_4\|u_{N,x}^\tau\|_\Lambda^2 + \frac{\beta^2}{2\epsilon_1} \sum_{j=1}^{M}\sum_{k=1}^{N} \gamma_k^j\|u_{h,x}^\tau\|_{\Lambda_k^j}^2.$$

The fact that $u_{h,x}^\tau$ is a piecewise constant and $u_N^\tau(\eta_{j,k}^c) = u_h^\tau(\eta_{j,k}^c)$, the third term of the left side of (5.1) can be estimated as

$$\sum_{j=1}^{M}\sum_{k=1}^{N} \gamma_k^j\,\beta^2\,(u_{N,x}^\tau, u_{h,x}^\tau)_{\Lambda_k^j} = \beta^2 \sum_{j=1}^{M}\sum_{k=1}^{N} \gamma_k^j\,u_{h,x}^\tau|_{\Lambda_k^j} \int_{\Lambda_k^j} u_{N,x}^\tau dx$$

$$(5.5) \qquad = \beta^2 \sum_{j=1}^{M} \sum_{k=1}^{N} \gamma_k^j \ u_{h,x}^{\tau}|_{\Lambda_k^j} \left(u_h^{\tau}(\eta_{j,k}^c) - u_h^{\tau}(\eta_{j,k-1}^c) \right)$$

$$= \beta^2 \sum_{j=1}^{M} \sum_{k=1}^{N} \gamma_k^j \| u_{h,x}^{\tau} \|_{\Lambda_k^j}^2 .$$

Finally, the terms of the right side of (5.1) can be done by

$$(5.6) \qquad \left| \sum_{j=1}^{M} \sum_{k=1}^{N} \gamma_k^j \, \beta \, (f, u_{h,x}^{\tau})_{\Lambda_k^j} \right| \leq \frac{\epsilon_2}{2} \frac{h^2}{12\nu} \|f\|_{\Lambda}^2 + \frac{\beta^2}{2\epsilon_2} \sum_{j=1}^{M} \sum_{k=1}^{N} \gamma_k^j \| u_{h,x}^{\tau} \|_{\Lambda_k^j}^2 ,$$

and

$$(5.7) \qquad |\, (f, u_N^{\tau})_{\Lambda}| \leq \frac{\epsilon_3}{2\nu} \|f\|_{\Lambda}^2 + \frac{\nu}{2\epsilon_3} \| u_N^{\tau} \|_{\Lambda}^2 .$$

Now choosing ϵ_1, ϵ_2 and ϵ_3 appropriately, and then combining all of these arguments yields the conclusion as done in [8]. $\qquad \square$

Lemma 5.2. *For all $\phi \in H^m(\Lambda)$, $m \geq 2$, there exists $\phi_N^{\tau} \in \mathcal{V}_N^{\tau}$ such that*

$$(5.8) \qquad \| \phi - \phi_N^{\tau} \|_{H^k(\Lambda)} \leq C_5 N^{k-m} \left(\frac{\tau}{2} \right)^{m-k} \| \phi \|_{H^m(\Lambda)}, \qquad k = 0, 1, 2,$$

where C_5 is a positive constant independent of N and mesh size τ.

Proof. The proof will be done by combining the proof of Proposition 3.2 and the classical result of Legendre spectral methods saying that there exists a polynomial $\hat{p}_N$ of degree N and in $H_0^1(\Lambda)$ such that

$$\| \hat{\phi}^j - \hat{p}_N^j \|_{H^k(\Lambda)} \leq C N^{k-m} \| \hat{\phi}^j \|_{H^m(\Lambda)}, \quad k = 0, 1, 2,$$

where $\hat{\phi}^j$ is the transformed reference function defined on Λ from the function ϕ defined on E_j. $\qquad \square$

Theorem 5.3. *Let u be the solution of (4.1), and let $u_N^{\tau} \in \mathcal{V}_N^{\tau}$ be the solution of (4.8). Under the same assumptions of Theorem 5.1, the following estimates holds*:

$$(5.9) \qquad \nu \| u - u_N^{\tau} \|_{H^1(\Lambda)} \quad \leq \quad C_6 N^{2-m} \left(\frac{\tau}{2} \right)^{m-2} \| u \|_{H^m(\Lambda)}$$

in which C_6 is a positive constant such that

$$C_6 := C \max \left\{ \frac{3\nu}{2} + \frac{h^2 \beta^2}{8\nu}, \frac{h^2 \nu}{8}, \frac{3\beta^2}{2\nu} \right\},$$

where the constant C only depends on the constant in Lemma 5.2.

Proof. Let u be the solution of (4.1), and let $u_N^{\tau} \in \mathcal{V}_N^{\tau}$ be the solution of (4.8). We want to estimate

$$u - u_N^{\tau} = (u - \bar{u}_N^{\tau}) + (\bar{u}_N^{\tau} - u_N^{\tau})$$

in the sense of Sobolev-norm. Here $\bar{u}_N^\tau \in \mathcal{V}_N^\tau$ is a polynomial according to Lemma 5.2. Set $w_N^\tau := u_N^\tau - \bar{u}_N^\tau \in \mathcal{V}_N^\tau$. Using (4.8), for all $v_N^\tau \in \mathcal{V}_N^\tau$, we have

$$
\nu(w_{N,x}^\tau, v_{N,x}^\tau)_\Lambda + \beta(w_{N,x}^\tau, v_N^\tau)_\Lambda
$$

$$
- \sum_{j=1}^M \sum_{k=1}^N \nu \gamma_k^j \beta(w_{N,xx}^\tau, v_{h,x}^\tau)_{\Lambda_k^j} + \sum_{j=1}^M \sum_{k=1}^N \gamma_k^j \beta^2(w_{N,x}^\tau, v_{h,x}^\tau)_{\Lambda_k^j}
$$

$$
(5.10) \qquad = \nu((u - \bar{u}_N^\tau)_x, v_{N,x}^\tau)_\Lambda + \beta((u - \bar{u}_N^\tau)_x, v_N^\tau)_\Lambda
$$

$$
+ \sum_{j=1}^M \sum_{k=1}^N \gamma_k^j(-\nu(u - \bar{u}_N^\tau)_{xx} + \beta(u - \bar{u}_N^\tau)_x, \beta v_{h,x}^\tau)_{\Lambda_k^j}.
$$

Let us choose $v_N^\tau = w_N^\tau$ in (5.10). Then one may get the lower bound for LHS of (5.10) using (5.4) with $\epsilon_1 = 1$ and (5.5) such that

$$
(5.11) \qquad \frac{1}{2}\left(\nu\|w_{N,x}^\tau\|_\Lambda^2 + \beta^2 \sum_{j=1}^M \sum_{k=1}^N \gamma_k^j \|w_{h,x}^\tau\|_{\Lambda_k^j}^2\right) \le \text{LHS of (5.10)}.
$$

The upper bound for RHS of (5.10) can be obtained by using Lemma 5.2. The first term of RHS of (5.10) can be estimated as:

$$
(5.12) \qquad \left|\nu((u - \bar{u}_N^\tau)_x, w_{N,x}^\tau)_\Lambda\right| \le \frac{3\nu}{2}\|u - \bar{u}_N^\tau\|_{H^1(\Lambda)}^2 + \frac{\nu}{6}\|w_{N,x}^\tau\|_\Lambda^2.
$$

The second term of RHS of (5.10) can be estimated as:

$$
\left|\beta((u - \bar{u}_N^\tau)_x, w_N^\tau)_\Lambda\right| = \left|\beta(u - \bar{u}_N^\tau, w_{N,x}^\tau)_\Lambda\right|
$$

$$
(5.13) \qquad\qquad\qquad \le \frac{3\beta^2}{2\nu}\|u - \bar{u}_N^\tau\|_\Lambda^2 + \frac{\nu}{6}\|w_{N,x}^\tau\|_\Lambda^2.
$$

The third term of RHS of (5.10) with help of Lemma 5.2 can be estimated as:

$$
\left|\sum_{j=1}^M \sum_{k=1}^N \gamma_k^j(-\nu(u - \bar{u}_N^\tau)_{xx} + \beta(u - \bar{u}_N^\tau)_x, \beta w_{h,x}^\tau)_{\Lambda_k^j}\right|
$$

$$
\le \sum_{j=1}^M \sum_{k=1}^N \gamma_k^j \nu\beta\|(u - \bar{u}_N^\tau)_{xx}\|_{\Lambda_k^j}\|w_{h,x}^\tau\|_{\Lambda_k^j}
$$

$$
+ \sum_{j=1}^M \sum_{k=1}^N \gamma_k^j \beta^2\|(u - \bar{u}_N^\tau)_x\|_{\Lambda_k^j}\|w_{h,x}^\tau\|_{\Lambda_k^j}
$$

$$
(5.14) \qquad \le \frac{h^2}{8}\nu\|u - \bar{u}_N^\tau\|_{H^2(\Lambda)}^2 + \frac{\beta^2}{6}\sum_{j=1}^M \sum_{k=1}^N \gamma_k^j\|w_{h,x}^\tau\|_{\Lambda_k^j}^2
$$

$$
+ \frac{h^2}{8\nu}\beta^2\|u - \bar{u}_N^\tau\|_{H^1(\Lambda)}^2 + \frac{\beta^2}{6}\sum_{j=1}^M \sum_{k=1}^N \gamma_k^j\|w_{h,x}^\tau\|_{\Lambda_k^j}^2.
$$

Then, combining (5.11)-(5.14) leads to

$$(5.15) \quad \frac{\nu}{6}\|w_{N,x}^\tau\|_\Lambda^2 + \frac{\beta^2}{6}\sum_{j=1}^{M}\sum_{k=1}^{N}\gamma_k^j\|w_{h,x}^\tau\|_{\Lambda_k^j}^2$$

$$\leq \left(\frac{3\nu}{2} + \frac{h^2\beta^2}{8\nu}\right)\|u - \bar{u}_N^\tau\|_{H^1(\Lambda)}^2 + \frac{h^2\nu}{8}\|u - \bar{u}_N^\tau\|_{H^2(\Lambda)}^2 + \frac{3\beta^2}{2\nu}\|u - \bar{u}_N^\tau\|_\Lambda^2$$

$$\leq C_7\|u - \bar{u}_N^\tau\|_{H^2(\Lambda)}^2,$$

where

$$C_7 := \max\left\{\frac{3\nu}{2} + \frac{h^2\beta^2}{8\nu}, \frac{h^2\nu}{8}, \frac{3\beta^2}{2\nu}\right\}.$$

Hence, it follows that, due to Lemma 5.2 and (5.15),

$$\nu\|u - u_N^\tau\|_{H^1(\Lambda)} \leq \nu\|u - \bar{u}_N^\tau\|_{H^1(\Lambda)} + \nu\|w_N^\tau\|_{H^1(\Lambda)}$$

$$\leq 6C_7\|u - \bar{u}_N^\tau\|_{H^2(\Lambda)}$$

$$\leq 6C_7C_5 N^{2-m}\left(\frac{\tau}{2}\right)^{m-2}\|u\|_{H^m(\Lambda)},$$

which implies the conclusion. □

For the numerical experiment, we consider the one-dimensional advection-diffusion problem

$$(5.16) \qquad\qquad -\nu u_{xx} + u_x = 1 \quad\text{in}\ \ \Lambda = (-1,1),$$
$$u(-1) = u(1) = 0$$

for some positive constant $\nu > 0$. Then we have the discretization formulation (4.8) with $\beta = 1$ and $f = 1$. In order to obtain the solvable system (4.8), we may determine the parameter value γ_k^j by choosing the appropriate bubble function b_k^j on Λ_k^j. But we have noticed the maximum value of γ_k^j is $\frac{(h_k^j)^2}{12\nu}$. So, we take the all values of γ_k^j into $\frac{(h_k^j)^2}{12\nu}$ for this numerical experiment. Even though these values are not optimal for the best approximated solution, it has shown that the numerical results with these parameter values are coincide with our theories by Tables 5.1, 5.2, and 5.3.

6. Lower-order preconditioning

Let us define the bilinear form $\Phi(\cdot,\cdot)$ on the space $\mathcal{V}_N^\tau \times \mathcal{V}_N^\tau$

$$(6.1) \qquad \Phi(u_N^\tau, v_N^\tau; v_h^\tau) := \nu(u_{N,x}^\tau, \overline{v}_{N,x}^\tau)_\Lambda + \Phi_c(u_N^\tau, v_N^\tau; v_h^\tau),$$

where the bilinear form $\Phi_c(\cdot,\cdot)$ on the space $\mathcal{V}_N^\tau \times \mathcal{V}_N^\tau$

$$(6.2) \qquad \Phi_c(u_N^\tau, v_N^\tau; v_h^\tau) := \beta(u_{N,x}^\tau, \overline{v}_N^\tau)_\Lambda - \sum_{j=1}^{M}\left[\sum_{k=1}^{N}\nu\,\gamma_k^j\beta(u_{N,xx}^\tau, \overline{v}_{h,x}^\tau)_{\Lambda_k^j}\right]$$

$$+ \sum_{j=1}^{M}\left[\sum_{k=1}^{N}\gamma_k^j\beta^2(u_{N,x}^\tau, \overline{v}_{h,x}^\tau)_{\Lambda_k^j}\right]$$

TABLE 5.1. H^1-norm error for $\nu = 0.1$

M\ N	2	4	8	16	32	64
2	.294385e+1	.889178e+0	.659017e-1	.165271e-1	.415315e-2	.103961e-2
4	.179261e+1	.208329e+0	.182148e-1	.464434e-2	.116685e-2	.292072e-3
8	.764476e+0	.295359e-1	.454696e-2	.115809e-2	.290883e-3	.728061e-4
16	.237980e+0	.420011e-2	.993903e-3	.252939e-3	.635187e-4	.158975e-4
32	.635988e-1	.751435e-3	.197609e-3	.502662e-4	.126214e-4	.315886e-5
64	.161819e-1	.139413e-3	.371579e-4	.944956e-5	.237254e-5	.594279e-6

TABLE 5.2. H^1-norm error for $\nu = 0.01$

M\ N	2	4	8	16	32	64
2	.139792e+2	.127978e+2	.774054e+1	.112373e+1	.478977e-1	.119863e-1
4	.136546e+2	.107593e+2	.394961e+1	.111396e+0	.160777e-1	.402462e-2
8	.126503e+2	.748412e+1	.109093e+1	.217056e-1	.544927e-2	.136414e-2
16	.103525e+2	.374312e+1	.111007e+0	.713149e-2	.179241e-2	.448692e-3
32	.689116e+1	.107147e+1	.906600e-2	.216932e-2	.545093e-3	.136448e-3
64	.331228e+1	.158956e+0	.226015e-2	.575819e-3	.144645e-3	.362045e-4

TABLE 5.3. H^1-norm error for $\nu = 0.001$

M\ N	2	4	8	16	32	64
2	.446994e+2	.446102e+2	.438123e+2	.370334e+2	.166229e+2	.777601e+0
4	.446785e+2	.444439e+2	.418371e+2	.281363e+2	.593263e+1	.433885e-1
8	.446371e+2	.438369e+2	.369007e+2	.165256e+2	.766581e+0	.144461e-1
16	.444732e+2	.418011e+2	.279408e+2	.585385e+1	.245651e-1	.506361e-2
32	.438299e+2	.366184e+2	.162383e+2	.769529e+0	.705939e-2	.176703e-2
64	.415806e+2	.273232e+2	.570497e+1	.205147e-1	.243198e-2	.608809e-3

and the bilinear form $\Phi_h(\cdot, \cdot)$ on the space $\mathcal{F}_h^\tau \times \mathcal{F}_h^\tau$

$$(6.3) \qquad\qquad \Phi_h(u_h^\tau, v_h^\tau) := \nu(u_{h,x}^\tau, \overline{v}_{h,x}^\tau)_\Lambda.$$

Define the linear form $F(\cdot)$ on the space $\mathcal{V}_N^\tau$

$$(6.4) \qquad F(v_N^\tau; v_h^\tau) := (f, \overline{v}_N^\tau)_\Lambda + \sum_{j=1}^{M}\Big[\sum_{k=1}^{N}\gamma_k^j \beta(f, \overline{v}_{h,x}^\tau)_{\Lambda_k^j}\Big].$$

For the implementation of (4.8) we need the linear system

$$(6.5) \qquad\qquad\qquad \mathsf{A}U = \mathsf{F},$$

where $\mathsf{A} := \Big(A_{\mu\kappa}\Big)$ and $\mathsf{F} := \Big(f_\kappa\Big)$ are

$$(6.6) \qquad A_{\mu\kappa} = \Phi(\phi_\mu, \phi_\kappa), \qquad f_\kappa = F(\phi_\kappa), \qquad \mu, \kappa = 1, 2, \ldots, MN-1,$$

Let us define the matrix $\mathsf{S} := \Big(S_{\mu\kappa}\Big)$ as

$$(6.7) \qquad S_{\mu\kappa} := \Phi_h(\psi_\mu, \psi_\kappa), \qquad \mu, \kappa = 1, 2, \ldots, MN-1.$$

With a nonzero vector $U = (u_1, \ldots, u_{MN-1})^T$, let

$$(6.8) \qquad u_N^\tau(x) := \sum_{\mu=1}^{MN-1} u_\mu \phi_\mu(x)$$

whose piecewise linear interpolation is denoted as

$$(6.9) \qquad u_h^\tau(x) := \mathcal{I}_N^\tau u_N^\tau(x) = \sum_{\mu=1}^{MN-1} u_\mu \psi(x).$$

Because of these interpolant relations, we can use the lower-order preconditioner S for A so that we will solve the equivalent preconditioned system

$$(6.10) \qquad \mathsf{S}^{-1}\mathsf{A}U = \mathsf{S}^{-1}\mathsf{F}.$$

Let

$$v_N^\tau(x) := p_N(x) + iq_N(x)$$

and let its piecewise linear interpolant $v_h^\tau(x) = \mathcal{I}_h^\tau v_N^\tau(x)$ be

$$v_h^\tau(x) = p_h(x) + iq_h(x).$$

Then, it follows that

$$
\begin{aligned}
Re\ \Phi(v_N^\tau, v_N^\tau; v_h^\tau) = {} & \nu |v_N^\tau|_{H^1(\Lambda)}^2 \\
& - \sum_{j=1}^{M}\sum_{k=1}^{N} \nu\gamma_k^j\beta \int_{\Lambda_k^j} p_N''(x)p_h'(x) + q_N''(x)q_h'(x)dx \\
& + \sum_{j=1}^{M}\sum_{k=1}^{N} \gamma_k^j\beta^2 \int_{\Lambda_k^j} p_N'(x)p_h'(x) + q_N'(x)q_h'(x)dx.
\end{aligned}
\tag{6.11}
$$

Lemma 6.1. *Assume that* (5.2) *holds. Then*

$$(6.12) \qquad Re\ \Phi(v_N^\tau, v_N^\tau; v_h^\tau) \geq \frac{\nu}{2}|v_N^\tau|_{H^1(\Lambda)}^2 + \frac{\beta^2}{2}\sum_{j=1}^{M}\sum_{k=1}^{N}\gamma_k^j\|v_{h,x}^\tau\|_{\Lambda_k^j}^2.$$

Proof. Following the same estimations as done in (5.4) and (5.5), it follows that

$$
\begin{aligned}
& \left| \sum_{j=1}^{M}\sum_{k=1}^{N} \nu\,\gamma_k^j\beta \int_{\Lambda_k^j} p_N''(x)p_h'(x) + q_N''(x)q_h'(x)dx \right| \\
& \leq \frac{\nu}{2}\max_{j,k} c(\hat{b}_k^j)C_4 |v_N^\tau|_{H^1(\Lambda)}^2 + \frac{\beta^2}{2}\sum_{j=1}^{M}\sum_{k=1}^{N}\gamma_k^j\|v_{h,x}^\tau\|_{\Lambda_k^j}^2 \\
& \leq \frac{\nu}{2}|v_N^\tau|_{H^1(\Lambda)}^2 + \frac{\beta^2}{2}\sum_{j=1}^{M}\sum_{k=1}^{N}\gamma_k^j\|v_{h,x}^\tau\|_{\Lambda_k^j}^2
\end{aligned}
\tag{6.13}
$$

and

$$(6.14) \quad \sum_{j=1}^{M}\sum_{k=1}^{N}\gamma_k^j\beta^2\int_{\Lambda_k^j} p_N'(x)p_h'(x) + q_N'(x)q_h'(x)dx = \beta^2\sum_{j=1}^{M}\sum_{k=1}^{N}\gamma_k^j\|v_{h,x}^\tau\|_{\Lambda_k^j}^2.$$

Then, the conclusion comes from combining (6.13), (6.14) with (6.11).   □

Lemma 6.2. *Assume that* (5.2) *holds. Then*

$$(6.15) \qquad \left| \Phi(v_N^\tau, v_N^\tau; v_h^\tau) \right| \leq \frac{3\nu}{2}|v_N^\tau|_{H^1(\Lambda)}^2 + \frac{3\beta^2}{2}\sum_{j=1}^{M}\sum_{k=1}^{N}\gamma_k^j\|v_{h,x}^\tau\|_{\Lambda_k^j}^2.$$

Further, it follows that

$$(6.16) \qquad \left| \Phi(v_N^\tau, v_N^\tau; v_h^\tau) \right| \leq \frac{3\nu}{2}|v_N^\tau|_{H^1(\Lambda)}^2 + C_8|v_h^\tau|_{H^1(\Lambda)}^2,$$

where

$$(6.17) \qquad C_8 := \frac{3\beta^2}{2\nu}\max_{1\leq j\leq M, 1\leq k\leq N} c(\hat{b}_k^j) = \frac{\beta^2}{8\nu}$$

which does not depend on N and τ.

Proof. The conclusion (6.15) holds by using (5.4) and (5.5). For the proof of (6.16), we have

$$(6.18) \quad \sum_{j=1}^{M}\sum_{k=1}^{N}\gamma_k^j\|v_{h,x}^\tau\|_{\Lambda_k^j}^2 \leq \max_{j,k} c(\hat{b}_k^j)\frac{1}{\nu}\sum_{j=1}^{M}\sum_{k=1}^{N}(h_k^j)^2\|v_{h,x}^\tau\|_{\Lambda_k^j}^2 = \frac{1}{12\nu}\|v_{h,x}^\tau\|_{\Lambda}^2.$$

Hence, we have the conclusion.   □

Now we have the following proposition.

Proposition 6.3. *Assume that* (5.2) *holds. Then it follows that*

$$(6.19) \qquad \frac{Re\ \Phi(v_N^\tau, v_N^\tau; v_h^\tau)}{\nu|v_h^\tau|_{H^1(\Lambda)}^2} \geq \frac{1}{2}C_2^2 > 0$$

and

$$(6.20) \qquad \frac{|\Phi(v_N^\tau, v_N^\tau; v_h^\tau)|}{\nu|v_h^\tau|_{H^1(\Lambda)}^2} \leq \frac{3}{2}C_3^2 + \frac{\beta^2}{8\nu^2}.$$

Proof. From Lemma 6.1 and Proposition 3.3, we have

$$\frac{Re\ \Phi(v_N^\tau, v_N^\tau; v_h^\tau)}{\nu|v_h^\tau|_{H^1(\Lambda)}^2} \geq \frac{\frac{\nu}{2}|v_N^\tau|_{H^1(\Lambda)}^2}{\nu|v_h^\tau|_{H^1(\Lambda)}^2} \geq \frac{1}{2}C_2^2 > 0,$$

and from (6.16) in Lemma 6.2 and Proposition 3.3, we have

$$\frac{|\Phi(v_N^\tau, v_N^\tau; v_h^\tau)|}{\nu|v_h^\tau|_{H^1(\Lambda)}^2} \leq \frac{\frac{3\nu}{2}|v_N^\tau|_{H^1(\Lambda)}^2}{\nu|v_h^\tau|_{H^1(\Lambda)}^2} + \frac{\frac{\beta^2}{8\nu}|v_h^\tau|_{H^1(\Lambda)}^2}{\nu|v_h^\tau|_{H^1(\Lambda)}^2} \leq \frac{3}{2}C_3^2 + \frac{\beta^2}{8\nu^2}.$$

Hence, we have the conclusion.   □

Theorem 6.4. *Assume that* (5.2) *holds. For any nonzero vector* U*, it follows that*

$$(6.21) \qquad \left| \frac{(AU, U)}{(SU, U)} \right| \ \le \ \Gamma \ < \ \infty, \qquad Re \ \frac{(AU, U)}{(SU, U)} \ \ge \ \gamma_0 \ > \ 0,$$

where $\Gamma := \frac{3}{2}C_3^2 + \frac{\beta^2}{8\nu^2}$ *and* $\gamma_0 = \frac{1}{2}C_2^2$ *are independent of the mesh size* τ *and the degree* N.

Proof. For a given nonzero vector U, let u_h^τ and u_N^τ be as (6.9) and (6.8) respectively. Then one has

$$\Phi_h(u_h^\tau, u_h^\tau) = \nu|u_h^\tau|_{H^1(\Lambda)}^2 = U^T SU,$$
$$\Phi(u_N^\tau, u_N^\tau) = U^T AU.$$

This argument completes the proof. □

For the numerical experiments for the eigenvalues of the preconditioned matrix, we use the one-dimensional advection-diffusion problem (5.16) same as in Section 5. We also take all values of γ_k^j into $\frac{(h_k^j)^2}{12\nu}$ for this numerical experiment. From Tables 6.1, 6.2, and 6.3, the numerical results reveal that the maximum absolute value of the eigenvalues of the preconditioned matrix increases in proportion to ν^{-2}, and the minimum of the real part of the eigenvalues is greater than $\gamma_0 = 1$ by Tables 6.4, 6.5, and 6.6.

TABLE 6.1. Maximum absolute value of the eigenvalues of the preconditioned matrix for $\nu = 0.1$

M\ N	2	4	8	16	32	64
2	.434374e+1	.365143e+1	.341003e+1	.335432e+1	.334090e+1	.333759e+1
4	.352792e+1	.340483e+1	.335405e+1	.334089e+1	.333759e+1	.333676e+1
8	.338027e+1	.335296e+1	.334082e+1	.333758e+1	.333676e+1	.333655e+1
16	.334717e+1	.334056e+1	.333757e+1	.333676e+1	.333655e+1	.333650e+1
32	.333914e+1	.333750e+1	.333675e+1	.333655e+1	.333650e+1	.333649e+1
64	.333715e+1	.333674e+1	.333655e+1	.333650e+1	.333649e+1	-

TABLE 6.2. Maximum absolute value of the eigenvalues of the preconditioned matrix for $\nu = 0.01$

M\ N	2	4	8	16	32	64
2	.211479e+3	.104521e+3	.367582e+2	.322802e+2	.319055e+2	.318583e+2
4	.618232e+2	.361711e+2	.322647e+2	.319052e+2	.318583e+2	.318494e+2
8	.347737e+2	.321963e+2	.319038e+2	.318582e+2	.318494e+2	.318474e+2
16	.321136e+2	.318976e+2	.318579e+2	.318494e+2	.318474e+2	.318469e+2
32	.318826e+2	.318572e+2	.318493e+2	.318474e+2	.318469e+2	.318467e+2
64	.318537e+2	.318492e+2	.318473e+2	.318469e+2	.318467e+2	-

Table 6.3. Maximum absolute value of the eigenvalues of the preconditioned matrix for $\nu = 0.001$

M\ N	2	4	8	16	32	64
2	.208365e+5	.104168e+5	.302827e+4	.765533e+3	.335911e+3	.319472e+3
4	.521896e+4	.260582e+4	.740998e+3	.335715e+3	.319469e+3	.318404e+3
8	.134137e+4	.658060e+3	.334921e+3	.319456e+3	.318403e+3	.318322e+3
16	.455997e+3	.331539e+3	.319403e+3	.318403e+3	.318322e+3	.318313e+3
32	.328797e+3	.319164e+3	.318399e+3	.318322e+3	.318313e+3	.318312e+3
64	.319025e+3	.318383e+3	.318322e+3	.318313e+3	.318312e+3	-

Table 6.4. Minimum of the real part of the eigenvalues of the preconditioned matrix for $\nu = 0.1$

M\ N	2	4	8	16	32	64
2	.325000e+1	.162969e+1	.123189e+1	.105861e+1	.101469e+1	.100368e+1
4	.157815e+1	.121962e+1	.105786e+1	.101465e+1	.100367e+1	.100092e+1
8	.114638e+1	.105477e+1	.101446e+1	.100366e+1	.100092e+1	.100023e+1
16	.103674e+1	.101368e+1	.100361e+1	.100092e+1	.100023e+1	.100006e+1
32	.100919e+1	.100342e+1	.100090e+1	.100023e+1	.100006e+1	.100001e+1
64	.100229e+1	.100086e+1	.100023e+1	.100006e+1	.100001e+1	-

Table 6.5. Minimum of the real part of the eigenvalues of the preconditioned matrix for $\nu = 0.01$

M\ N	2	4	8	16	32	64
2	.209500e+3	.204928e+2	.361604e+1	.161745e+1	.133471e+1	.124324e+1
4	.531409e+2	.690722e+1	.201112e+1	.138112e+1	.128049e+1	.108372e+1
8	.140371e+2	.303637e+1	.149318e+1	.130540e+1	.108365e+1	.102093e+1
16	.425944e+1	.179294e+1	.132851e+1	.108335e+1	.102091e+1	.100523e+1
32	.181487e+1	.130853e+1	.108213e+1	.102084e+1	.100523e+1	.100131e+1
64	.120372e+1	.107713e+1	.102053e+1	.100521e+1	.100131e+1	-

Table 6.6. Minimum of the real part of the eigenvalues of the preconditioned matrix for $\nu = 0.001$

M\ N	2	4	8	16	32	64
2	.208345e+5	.178989e+4	.124599e+3	.121016e+2	.293977e+1	.163712e+1
4	.520939e+4	.449477e+3	.340065e+2	.515369e+1	.195353e+1	.141762e+1
8	.130309e+4	.114368e+3	.111259e+2	.267839e+1	.152776e+1	.132247e+1
16	.326525e+3	.305662e+2	.462662e+1	.176582e+1	.136140e+1	.126102e+1
32	.823813e+2	.950922e+1	.235054e+1	.143058e+1	.129571e+1	.113069e+1
64	.213453e+2	.385557e+1	.159901e+1	.132169e+1	.113057e+1	-

7. Conclusion

The Chebyshev-Legendre high-order methods for a bubble stabilized Galerkin scheme is analyzed in solving advection-diffusion equations. The H^1 norm

convergence is shown in terms of the viscosity constant, mesh sizes and degrees of Chebyshev polynomials. Further the finite element lower-order preconditioner is completely analyzed for the proposed stabilized scheme. As shown by computational results, one may understand that the theory proved for the proposed scheme is suitable. In particular, it shows that the eigenvalues are independent of mesh sizes and degrees of polynomials. If one wants to compute the proposed scheme for a given advection-diffusion equation, one may use the derivative matrix in [10] for a computation of (6.5) which uses CGL and LGL points.

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